Int. J. Heat Mass Transfer. Vol. 19, 707–714. Pergamon Press 1976. Printed in Great Britain

TRANSIENT HEAT CONDUCTION IN CYLINDRICAL SYSTEMS WITH AN AXIALLY MOVING BOUNDARY

J. KERN and J. O. HANSEN
Department of Chemical Engineering, University of the Witwatersrand,
Jan Smuts Ave., Johannesburg, South Africa

(*Received* 23 *July* 1974)

Abstract—A hot granular material is cooled already during filling into a cylindrical container. The rate of heat lost through the shell depends on the materials and temperatures involved, the geometry of the system as well as the filling rate. Due to the moving boundary there is no straightforward solution for the temperature distribution available. However, from a model based on the principle of quasi-stationary state the charge temperature and the rate of heat loss can be calculated. The theory is applicable also to viscous liquids with negligible internal convection. The results agree well with experimental findings and two industrial applications are discussed.

NOMENCLATURE

D_p, average size of granulate [m];
h, heat-transfer coefficient [$Wm^{-2}K^{-1}$];
H, total height of silo [m];
k, thermal conductivity [$Wm^{-1}K^{-1}$];
q, heat flux [kW];
Q, total heat loss [kWh];
r, radial coordinate [m];
R, radius of inside shell surface [m];
t, time [s] or [h];
Δt, total filling time [h];
T, temperature [°C];
T_f^*, genuine filling temperature [°C];
U, overall heat-transfer coefficient at the wall [$Wm^{-2}K^{-1}$];
v, filling velocity [ms^{-1}];
V, silo capacity [m^3];
$z, \bar{z}$, axial coordinate in moving, fixed coordinate system [m];
$\bar{z}_0$, filling height from bottom to cone base [m];
α, thermal diffusivity [m^2s^{-1}];
Δ, shell thickness [m];
$\bar{\rho}$, mean density of granulate [kgm^{-3}].

Dimensionless quantities

A, $= vR/\alpha$, velocity parameter;
B, coefficient;
Bi, $= UR/k$, Biot number;
C_1, C_2, constants;
Fo, $= t\alpha/R^2$, Fourier number;
J_0, J_1, Bessel functions;
N, number of time intervals;
δ, $= z\alpha/(vR^2)$, depth from cone base;
ζ, $= zv/\alpha$, axial coordinate;
θ, $= (T-T_a)/(T_f-T_a)$, temperature;
λ, characteristic value;
ρ, $= r/R$, radius.

Subscripts

a, ambient;
f, filling;
i, j, n, general integer;
w, wall.

1. INTRODUCTION

VARIOUS process schemes in chemical engineering involve a temporary storage of material in cylindrical containers or silos. The main purpose is either to smooth a fluctuating flow rate between two consecutive processes or to have a buffer volume available in case one of the earlier process stages breaks down. Usually two containers in parallel are provided so that material can be drawn from one of them while the other one is being filled. For small and short fluctuations a single container may be adequate. As most processes are operated above surrounding temperature the largest rate of cooling will occur during filling. Depending on the requirements of the following process stage this effect may be desirable or not. In the former case the storage device should simultaneously be designed as an effective cooler whereas in the latter situation one would try and minimize the heat loss.

An additional problem arises with the structural design of large silos. Apart from static stresses one has to consider thermal stresses due to different temperatures on either side of the shell. At any fixed height below the filling level the two stresses change with time, but in different directions. While the static stress builds up due to more material being added at the top, the thermal stress decreases due to heat being lost through the shell. The sum of both stresses will be a maximum at some distance below the top, usually not at the bottom where the static stress is highest. To our knowledge this conclusion is not incorporated in common design practice because too little is known about the temperature distribution developing in a silo. With pre-stressed concrete silos, commonly used to store hot cement clinker and similar materials, the stress distribution is more complicated, but here as well the knowledge of the inside wall temperature as a function of time and position is of great importance for an economical design of the silo. Experiments [1] have shown that the described effect is marked but because of the small number of results no general conclusions could be drawn. Therefore a more fundamental study had to be undertaken, and the present

analysis will disclose all the relevant parameters and show how they affect the temperature distribution.

One of the obvious parameters is the volumetric filling rate of material. For a given geometry there results a certain velocity at which the filling level will move away from the bottom of the container. Writing down the differential equation in a fixed coordinate system we get a two-dimensional unsteady conduction problem with a moving boundary. An attempt to solve this problem analytically failed and from a study of relevant sources [2, 3] it seems that a straightforward solution with applicable results is not possible at present. The idea then is to let the coordinate system move with the filling level whereby the time-dependency is eliminated and the velocity of the moving boundary enters the problem as a parameter. The basic concept of this method was put forward as a result of experimental studies on arc-welding [4], where a point source of heat moves along the longer axis of the system. It was observed that the temperature distribution around the moving source becomes independent of time when the system axis is long enough in comparison with the energy dissipated. This implies that any reflection from the boundaries of the system due to rapid penetration of heat can be neglected. Hence the system does not need to be long in a geometrical sense. From the viewpoint of an observer moving with the point source there results a steady-state conduction problem which for obvious reasons is simpler to solve.

On the basis of this so-called principle of quasi-stationary state the general theory of moving heat sources was analyzed in detail [5] but mainly with the application to welding processes in mind. For this reason the analytical results cannot be applied to the problem described here. The only case of a cylindrical system being studied was that of the temperature distribution generated in a gun barrel by firing a bullet [6]. However, due to the boundary conditions this problem is mathematically completely different from ours although they both employ the same principle.

As a consequence we find it necessary to develop the solution for the outlined class of problems from the basic differential equation. Physical modelling will be used frequently in order to obtain simple and applicable results. The latter are illustrated in the discussion of two common industrial applications.

2. PROBLEM AND MATHEMATICAL SOLUTION

In order to stress the main features of the method we restrict ourselves to the treatment of granular material and do not consider the storage of liquids. It turns out that the latter can be analyzed in the same way but more easily because of the simpler geometry. Referring to Fig. 1 we are dealing with a cylindrical shell of up to 30 m in diameter and 50 m in height into which the material is continuously filled over a period of up to 25 days. The physical properties of the charge are taken as independent of the temperature and we treat the stored material as a continuum using the mean properties throughout. This is justified by the

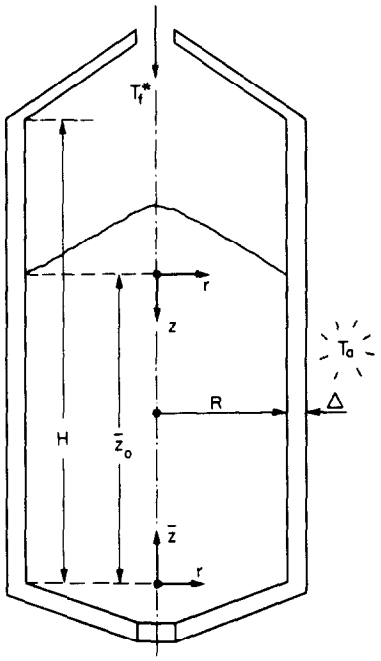

FIG. 1. Silo geometry.

fact that the silo diameter is much larger than the average particle diameter of the granulate. It is further assumed that the temperature does not vary with angular position but only with radius and height.

2.1. *Direct analysis*

Under these assumptions the temperature in a volume element of stored material is described by the Fourier equation

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial \bar{z}^2}\right), \tag{1}$$

where r and $\bar{z}$ are the coordinates shown in Fig. 1. The boundary conditions on r express the symmetry in the centre and the continuity of heat flux at the wall

$$\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0 \tag{2}$$

and

$$-k\left.\frac{\partial T}{\partial r}\right|_{r=R} = U(T|_{r=R} - T_a). \tag{3}$$

Here k is the effective thermal conductivity of the granulate for which correlations are available elsewhere [7]. U is the overall heat-transfer coefficient on the boundary which for a packed bed can be written as

$$U = \left(\frac{D_p}{k} + \frac{\Delta}{k_w} + \frac{1}{h_a}\right)^{-1}. \tag{4}$$

It has been shown [8] that in the region of half a particle diameter from the wall the effective conductivity drops to about half its bulk value.

The boundary conditions on $\bar{z}$ are not as easily specified. At the surface of the filling cone the material will usually be at filling temperature T_f^* because the air above the cone will soon be close to T_f^* as well.

However, every material develops a different cone shape and in order to keep the number of parameters small we reduce the constant temperature on the cone surface to a temperature profile at constant height $\bar{z}_0$ slightly below the circular base of the cone. Obviously a particle near the wall is cooled faster than one in the central region. The latter, on the other hand, has spent more time in the silo until it reaches the same absolute height as the wall particle. Depending on the thermal properties of the system there will consequently develop a certain temperature profile at $\bar{z}_0$. This model is illustrated in Fig. 2 and in order to simplify the analysis a constant temperature T_f at $\bar{z}_0$ is assumed.

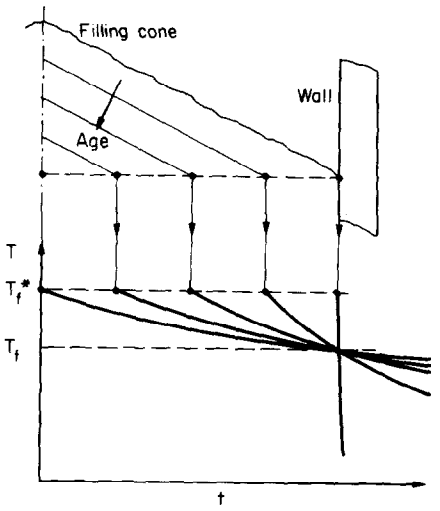

FIG. 2. Model for elimination of filling cone.

As is shown later any other profile could be used as well. The boundary condition becomes

$$T|_{\bar{z}=\bar{z}_0(t)} = T_f = \text{constant}. \tag{5}$$

The other condition at $\bar{z} = 0$ as well as the initial condition are not discussed here because they will take a different form in the final analysis. As can be seen from Fig. 1 their specification would need some more simplifying assumptions. However, it is concluded from equations (1) and (5) that even if a two-dimensional series solution were obtained the coefficients become time-dependent because they have to be evaluated from equation (5). Due to this complication the direct way of solving the problem had to be abandoned. It remains doubtful whether a straightforward analytical solution is at all possible without further simplification.

2.2. *Model*

Following the introductory discussion we now move the coordinate system into the plane $\bar{z}_0(t)$ which is indicated in Fig. 1. With the filling cone being eliminated in the same way as before the upper boundary surface at T_f may be seen as a source of heat such that the material in the source plane undergoes a step change to T_f. This rigid boundary condition effectively insulates the behaviour in the solid bulk of the silo from that in the only theoretically existent section above the source plane. The origin of the new coordinate system moves upwards at the filling velocity v which generates the following transformation equation

$$z = vt - \bar{z}. \tag{6}$$

The new axial coordinate is a combination of the two previous variables $\bar{z}$ and t and can be interpreted as the age of a particle in the silo or, simultaneously, its depth from the surface. Apparently z is undefined for $vt < \bar{z}$, which would refer to a particle not yet added to the silo. Rewriting the time derivative of the temperature for a point at radius r and depth z we obtain

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + v\frac{\partial T}{\partial z}. \tag{7}$$

With the assumption that the temperature in the new coordinate system is independent of time equation (1) changes to

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} - \frac{v}{\alpha}\frac{\partial T}{\partial z} = 0. \tag{8}$$

Hence, the quasi-stationary state, referred to earlier, is mathematically expressed by

$$\frac{\partial T}{\partial t} = 0.$$

The conditions under which this assumption holds have been discussed before and it seems that for the applications in mind the quasi-stationary state is justified. Only for an extremely small filling velocity together with a very high thermal diffusivity of the charge the results will be of an approximate character.

The boundary conditions on r remain unchanged whereas those on z now become simpler than before. At the top we have

$$T|_{z=0,r} = T_f = \text{constant} \tag{5a}$$

and the model automatically fixes the second boundary condition

$$T|_{z\to\infty,r} = \text{constant} = T_a. \tag{9}$$

Firstly it is necessary to treat the stored material as a semi-infinite cylinder or else we would not comply with the assumption of no reflection. But apart from that equation (8) cannot be solved with a boundary condition that specifies the temperature at the time-dependent depth $\bar{z}_0(t)$. Consequently the solution will be an approximation for the early stage of filling. The accuracy obviously depends on the system parameters but a qualitative estimate indicates that the approximate nature of the results may range up to a filling height of only 20%. The effect of the simplification will in most cases slightly under-represent the heat loss during filling, yet will hardly affect the determination of critical stresses in the shell. These conclusions should be borne in mind whenever a high accuracy in the results is required.

2.3. *General solution*

In order to develop the results in a most general form the following dimensionless variables are chosen:

$$\theta = \frac{T_{(r,z)} - T_a}{T_f - T_a}; \quad \rho = \frac{r}{R}, \quad \zeta = \frac{z \cdot v}{\alpha}.$$

The differential equation thereby becomes

$$\frac{\partial^2\theta}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\theta}{\partial\rho}+A^2\left(\frac{\partial^2\theta}{\partial\zeta^2}-\frac{\partial\theta}{\partial\zeta}\right)=0, \tag{8a}$$

where the dimensionless parameter $A = vR/\alpha$ is introduced. The boundary conditions are

$$\left.\frac{\partial\theta}{\partial\rho}\right|_{\rho=0,\zeta}=0, \tag{2a}$$

$$\left.\frac{\partial\theta}{\partial\rho}\right|_{\rho=1,\zeta}=-Bi\theta, \tag{3a}$$

$$\theta|_{\rho,\zeta}=1 \tag{5b}$$

and

$$\theta|_{\rho,\zeta\to\infty}=0. \tag{9a}$$

This system can be solved by separation of variables and the details of the solution are found in the Appendix. After insertion of the four boundary conditions one obtains

$$\theta(\rho,\zeta)=2\sum_{n=1}^{\infty}\left\{\frac{BiJ_0(\lambda_n\rho)}{(\lambda_n^2+Bi^2)J_0(\lambda_n)}\times\exp[(1-\sqrt{[1+4\lambda_n^2/A^2]})\zeta/2]\right\}. \tag{10}$$

The characteristic values λ_n are the roots of the transcendental equation

$$\frac{\lambda_nJ_1(\lambda_n)}{J_0(\lambda_n)}-Bi=0. \tag{11}$$

The temperature distribution in the silo thus depends on the Biot number and a velocity parameter A. However, for practical applications this latter parameter can be included in the axial coordinate so that the solution becomes simpler. With the low thermal diffusivity of all granular materials the value of A will usually lie between 10^2 and 10^3. Then the following binomial approximation is justified:

$$[1-\sqrt{(1+4\lambda_n^2/A^2)}]\simeq-2\lambda_n^2/A^2.$$

Introducing a new dimensionless depth $\delta = z\cdot\alpha/(vR^2)$ we rewrite equation (10) to get

$$\theta(\rho,\delta)=2\sum_{n=1}^{\infty}\left[\frac{BiJ_0(\lambda_n\rho)}{(\lambda_n^2+Bi^2)J_0(\lambda_n)}\exp(-\lambda_n^2\delta)\right]. \tag{12}$$

RESULTS

The nature of δ indicates that, with an approximation, the effect of doubling the filling velocity is to double the depth at which a certain temperature is observed. This is not surprising because it is merely the time spent in the silo which has been conserved. The results obtained from equation (10) show that with the achieved accuracy on the characteristic values λ_n velocity and depth are perfectly interchangeable.

As the filling velocity is assumed to be constant the new variable describes how at a certain time the temperature changes with depth. Simultaneously this variable determines the temperature change with time at a certain absolute height. With

$$z/v = t$$

we define the Fourier number as

$$Fo\equiv\delta=\frac{t\cdot\alpha}{R^2}$$

and equation (12) is represented as

$$\theta(\rho,Fo)|_{\Xi_1}=2\sum_{n=1}^{\infty}\left[\frac{BiJ_0(\lambda_n\rho)}{(\lambda_n^2+Bi^2)J_0(\lambda_n)}\exp(-\lambda_n^2Fo)\right]. \tag{12a}$$

Here Ξ_1 is an absolute height, measured from the bottom of the silo and equation (12a) is valid from the time when the charge, at temperature T_f, passes the level Ξ_1.

Although the form of equation (12) is analytic it may be useful to discuss some of the important results. The major problem encountered in the computation was the extraction of the roots λ_n. The convergence of the series in equation (10) is dictated largely by the exponential term. As δ usually is a small quantity up to 50 roots have to be evaluated from equation (11). This ensures that at a depth of one radius the calculated temperature is accurate to 0.5%. Using a Newton–Raphson search procedure the λ_n can be evaluated for different Biot numbers. However, the function $\lambda_nJ_1(\lambda_n)/J_0(\lambda_n)$ has discontinuities at the zero values of $J_0(\lambda_n)$ so that it becomes difficult to evaluate the roots at Biot numbers larger than 40. If these were ever needed for a practical problem, one would have to adjust the accuracy criterion.

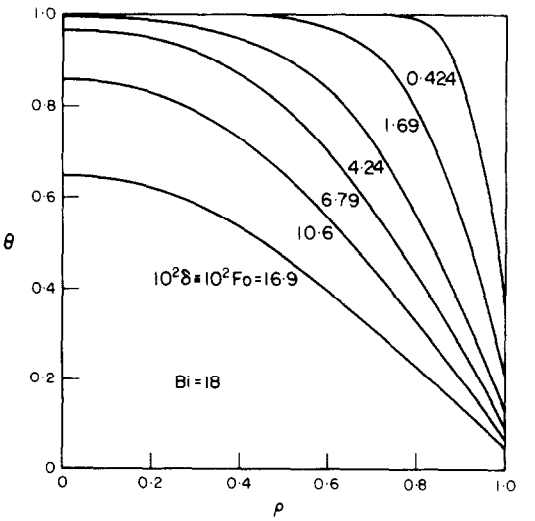

FIG. 3. Radial temperature distribution at variable depth or time.

The radial temperature distribution at various depths and for a typical Biot number of 18 is shown in Fig. 3. One realizes that only a large distance from the filling level does the temperature in the centre start to drop. This agrees with the observation [1] that some of the material drawn from a silo two weeks after filling is still at the original filling temperature. As explained previously this figure also represents the radial temperature distribution at a certain height for various time intervals.

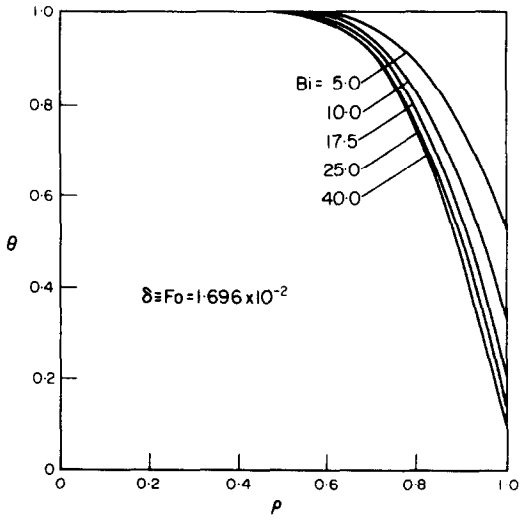

FIG. 4. Radial temperature distribution at various ratios of bulk to wall resistance.

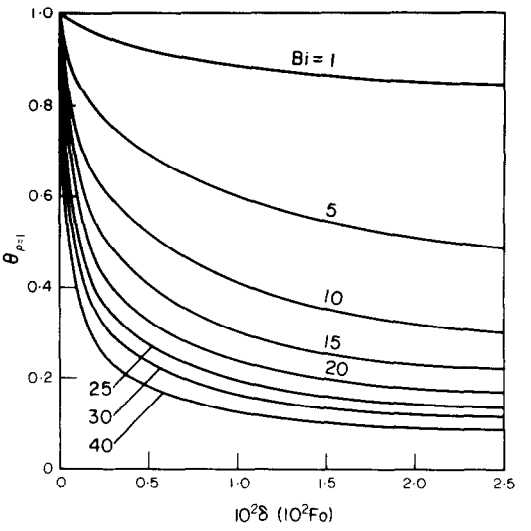

FIG. 5. Variation of inside wall temperature with time or depth for different Biot numbers.

The influence of the Biot number on the temperature profile is illustrated in Fig. 4, where a constant dimensionless depth $\delta = 0.01696$ was chosen. Due to the low thermal conductivity of the charge a variable wall transfer coefficient U affects mainly the material close to the wall, which is physically reasonable. In particular the wall temperature itself changes markedly and these results may help in an optimum structural design of a silo.

From a practical point of view only the knowledge of the wall temperatures is really interesting. For a given value of U it allows the calculation of heat losses as well as thermal stresses. Therefore Fig. 5 was constructed which shows the inside wall temperature at variable depth or time and for different heat loss characteristics. It is seen that especially a short time after the filling level has passed a certain position the wall temperature here decreases sharply with increasing Biot number. Once these curves are available it is simple to optimize the thermal or structural design of a silo and to illustrate this two different examples will be discussed.

4. APPLICATIONS

As pointed out in the beginning the purpose of this study was to provide the background for an economic design of a silo. The actual problem may lie in the structural design, where the wall thickness is determined by the stresses to be taken up, or in the thermal behaviour, where, apart from the storage, simultaneous cooling may be advantageous or not. It is therefore obvious that we look at the developed solution in the light of these two features.

4.1. *Inside wall temperature in a reinforced concrete silo*

As a detailed stress analysis is too involved to be presented here, we restrict the treatment to the comparison of our analytical results with measurements taken on a reinforced concrete silo during filling. Generally the moment developed in the wall of a large cylindrical container is linearly proportional to the temperature difference between the inside and outside wall surface. As this moment has to be taken up by the shell its thickness will depend on the inside wall temperature. If the latter is estimated too conservatively a double effect results. Firstly, the shell thickness becomes larger than necessary and secondly, as the thermal resistance of the wall has a major influence on U, a thicker wall gives an even higher wall temperature which again builds up the moment. Therefore a reasonably accurate information on the wall temperature distribution is important.

Unfortunately in the experiment [1] some of the relevant data are missing and have to be estimated. They are marked in Table 1 which shows the completed set of data needed for the comparison. The measuring points were at a height of 20 m from the bottom of the silo and the wall temperatures were measured in the concrete at 0.035 m from the outside and 0.03 m from the inside surface. Hence the reported inside wall

Table 1. Silo data for different applications (* = estimated)

	R	H	$10^{-3}V$	k	$10^{-3}\bar{\rho}$	$10^{6}\alpha$	U	Δt	$10^{2}Fo_{\Delta t}$	Bi	$(T_f - T_a)$	Q_t†	Q_t‡
Silo [1]	15.9	41.9	33.3	0.9*	1.58	0.8	1.8*	336	0.38	30	41.5*	—	—
Silo I	2	40	0.503	1.0	1.6	1.0	5.0	24	2.16	10	100.0	1535	1680
Silo II	4	10	0.503	1.0	1.6	1.0	5.0	24	0.54	20	100.0	768	865

†Analytical result from equation (14a).
‡Numerical result from equation (16).

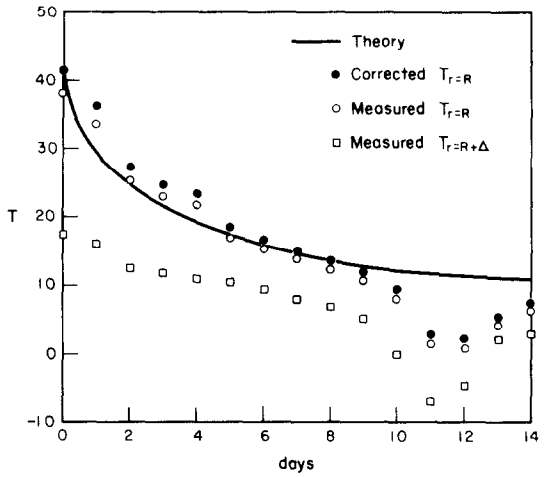

FIG. 6. Comparison between experiment [1] and theory.

temperature has to be extrapolated to the radius used in the analysis. This is done by employing equation (4) and is illustrated in Fig. 6. No record was kept on the ambient temperature but the measured temperature profiles indicate an ambient temperature of 0°C in the first six days followed by a slight drop and after nine days another sharp decrease. The time intervals in this case refer to the time when the filling level passes the height of the measuring points. From the definition of the dimensionless temperature it is seen that an incorrect ambient temperature can introduce a large error in the results. Especially towards the end of the filling period where the wall temperature is low a change of 5°C in the outside temperature may easily result in a 30% error in the calculated wall temperature based on the original outside temperature. Apparently this effect is responsible for the poor agreement between experiment and theory at large time as illustrated in Fig. 6. However, the tendency is represented fairly well by the theory and it is unfortunate that no other measurements are reported in the literature. The slightly higher wall temperatures measured in the beginning are most probably due to the very large radius of the experimental silo. At time = 0 the material temperature in the centre was certainly higher than near the wall, so that cooling was partly offset by the higher internal energy of the particles in the centre. In addition, the temperature at zero time may have been larger than the measured one, because measurements were carried out only once a day.

4.2. *Heat loss from a buffer silo*

For the same feed rate and total volume to be stored there are various ways to influence the heat loss during filling. Changing the shell material or splitting up the capacity into two or more silos are possible means, but one simple way is to change the height to diameter ratio. This latter procedure is illustrated in the following example for which the relevant data are listed in Table 1. Both silos are made from steel so that the wall-transfer coefficient U is appreciably larger than for a concrete shell. Height and diameter are changed to result in the same capacity for both units.

With equation (6) the re-transformation of equation (12) into the fixed coordinate system yields

$$\theta(\rho, \bar{z}) = 2 \sum_{n=1}^{\infty} \left[\frac{BiJ_0(\lambda_n \rho)}{(\lambda_n^2 + Bi^2)J_0(\lambda_n)} \times \exp(-\lambda_n^2 \alpha \cdot t/R^2) \exp(\lambda_n^2 \alpha \cdot \bar{z}/(vR^2)) \right]. \quad (13)$$

In real variables the total heat loss is then obtained from the following equation:

$$Q_t = 2\pi R \int_0^{\Delta t} \int_0^{z=vt} \left(-k \frac{\partial T}{\partial r}\bigg|_{r=R} \right) \mathrm{d}\bar{z}\,\mathrm{d}t, \quad (14)$$

and with equation (13) one finally gets

$$Q_t = 4V\rho C_p(T_f - T_a) \times \sum_{n=1}^{\infty} \left[\frac{1-(1-\exp(-\lambda_n^2 Fo_{\Delta t}))/(Fo_{\Delta t}\lambda_n^2)}{\lambda_n^2((\lambda_n/Bi)^2+1)} \right]. \quad (14a)$$

Once the series is evaluated for various combinations of the Biot and Fourier number this result allows one to immediately determine the cup mixing temperature of the charge after filling. However, the characteristic values λ_n depend on the Biot number (see equation (11)) and the accuracy of equation (14a) depends strongly on a correct determination of the λ_n. In addition, the convergence of the series in equation (14a) is even poorer than that in equation (12) so that one requires a large number of characteristic values. Their evaluation becomes tedious for the reasons discussed before and an alternative method is suggested which is based on the availability of the curves in Fig. 5.

For a differential area the heat loss is given by

$$\mathrm{d}q = U\theta(T_f - T_a)2\pi R\,\mathrm{d}\bar{z}. \quad (15)$$

Once the curves in Fig. 5 are known the integration of equation (15) is most conveniently done numerically. Subdividing the filling period into N time intervals the corresponding average values $\bar{\theta}$ are taken from the curve for the specific Biot number. During the first time interval which also determines the first area element the heat flux becomes

$$\Delta q_1 = U\bar{\theta}_1(T_f - T_a)\left(\frac{2\pi RH}{N}\right). \quad (15a)$$

In every following time interval the lower heat flux from the silo element below has to be added to equation (15a). For the data in Table 1 this procedure is illustrated in Fig. 7. The dotted lines near the origin indicate that in the beginning of the filling period our analysis predicts too little heat loss. Under most circumstances some energy is lost through the bottom, but usually this contribution becomes negligible after a short time. After filling has been completed the total heat flux from the silo drops sharply, which is the reason for concentrating the effort on the filling period.

Multiplying equation (15a) by the time interval and summing up over the filling period we get the total heat loss

$$Q = Bik(T_f - T_a)\frac{2\pi H^2}{vN^2} \sum_{i=1}^{N} [(N-i+1)\bar{\theta}_i]. \quad (16)$$

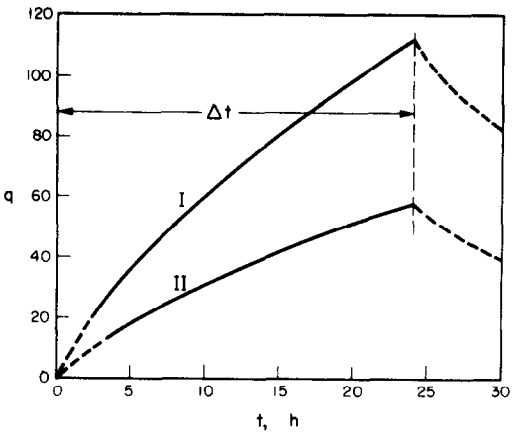

FIG. 7. Heat loss from a silo during filling.

If the required accuracy is not too high then equation (16) enables one to do a quick estimate on the heat loss to be expected. Choosing $N = 12$ for the two silo configurations we get a total heat loss of 1680 kWh for the high silo (I) as compared to 865 kWh for the wide one (II). The corresponding results from equation (14a) are displayed in Table 1 and are approximately 10% lower. This is explained by the fact that the numerical integration only uses the arithmetic average value of the temperature in the time interval chosen. For higher accuracy one would therefore have to decide on smaller time intervals.

An interesting conclusion from equation (14a) is that doubling the silo shell area yields twice the heat loss during filling. The result was checked for various other combinations of diameter and height and proved to be true although we could not find a straightforward physical explanation for this effect. Apparently the temperature profiles developing in the two silos I and II are different, but this is probably compensated by the different volumes of material undergoing a temperature change. However, as this example was only meant to illustrate the developed method, the results were not followed up any further.

5. CONCLUSION

From the discussed examples it is evident that the temperature distribution in a charge being filled into a cylindrical container can supply valuable information about thermal stresses in the shell or heat loss from the container. Assuming a pure conduction mechanism there remains the problem of incorporating in the analysis the moving boundary at the top of the charge. This is circumvented by taking a moving coordinate system and employing the principle of quasi-stationary state established elsewhere. With a further simplification a solution is obtained for the radial temperature distribution as a function of depth from the surface or time and with the Biot number as a parameter. For practical application the inside wall temperature of a silo usually is the most relevant information. The latter is provided by a single set of curves.

The study was carried out with a granular charge in mind but the results also apply to liquids as long as internal convection can be neglected. If this is not the case the analysis will still supply approximate results. An important application, not discussed here, is the storage of solutions where undesirable crystallization starts below a specific temperature. In general, it is concluded that this method could efficiently be applied to a large number of problems in chemical engineering processes.

REFERENCES

1. H. J. Klischat, Temperaturmessungen an einem Stahlbeton-Klinkersilo, *Zement-Kalk-Gips* **10**, 494–495 (1972).
2. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, 2nd edn. Clarendon Press, Oxford (1959).
3. V. S. Arpaci, *Conduction Heat Transfer*. Addison-Wesley, Palo Alto (1966).
4. H. Bornefeld, Temperaturmessungen beim Schweissen, *TZprakt. Metallbearb.* **43**, 14–18 (1933).
5. D. Rosenthal, The theory of moving sources of heat and its application to metal treatments, *Trans. Am. Soc. Mech. Engrs* **68**, 849–866 (1946).
6. D. Rosenthal and R. H. Cameron, Temperature distribution in cylinder heated by point source moving along its axis, *Trans. Am. Soc. Mech. Engrs* **69**, 961–968 (1947).
7. S. Yagi and D. Kunii, Studies on effective thermal conductivities in packed beds, *A.I.Ch.E. Jl* **3**, 373 (1957).
8. K. Ofuchi and D. Kunii, Heat-transfer characteristics of packed beds with stagnant fluids, *Int. J. Heat Mass Transfer* **8**, 749–757 (1965).
9. R. V. Churchill, *Operational Mathematics*, 3rd edn, p. 425. McGraw-Hill, New York (1972).

APPENDIX

Solution of the Differential Equation

Separation of variables yields two ordinary differential equations instead of equation (8a). With

$$\theta(\rho, \zeta) = P(\rho)Z(\zeta) \tag{A1}$$

we get

$$\rho \frac{d^2P}{d\rho^2} + \frac{dP}{d\rho} + \lambda^2 \rho P = 0 \tag{A2}$$

and

$$A^2 \frac{d^2Z}{d\zeta^2} - A^2 \frac{dZ}{d\zeta} - \lambda^2 Z = 0. \tag{A3}$$

These are solved to give

$$P = \sum_{n=1}^{\infty} B_n J_0(\lambda_n \rho) \tag{A4}$$

and

$$Z = C_1 \exp\{[1 + \sqrt{(1 + 4\lambda_n^2/A^2)}]\zeta/2\} + C_2 \exp\{[1 - \sqrt{(1 + 4\lambda_n^2/A^2)}]\zeta/2\}. \tag{A5}$$

From the boundary condition equation (9a) it follows immediately that $C_1 = 0$. Instead of considering the other condition equation (5b) directly, we set $C_2 = 1$ and adjust equation (A4) to satisfy the source plane boundary condition. Equation (A1) thereby becomes

$$\theta(\rho, \zeta) = \sum_{n=1}^{\infty} \{B_n J_0(\lambda_n \rho) \exp[(1 - \sqrt{(1 + 4\lambda_n/A^2)})\zeta/2]\}. \tag{A6}$$

With

$$\frac{dJ_0(\lambda_n \rho)}{d\rho} = -\lambda_n J_1(\lambda_n \rho)$$

it is seen that the Bessel solution intrinsically satisfies the boundary condition equation (2a) because $J_1(0) = 0$.

Inserting further equation (A6) into equation (3a) we obtain

$$\sum_{n=1}^{\infty} [B_n \lambda_n J_1(\lambda_n) Z_n] = Bi \sum_{n=1}^{\infty} [B_n J_0(\lambda_n) Z_n]. \quad \text{(A7)}$$

Then by matching terms in the two series and dividing by the common Z_n-terms one finds

$$\lambda_n J_1(\lambda_n) = Bi J_0(\lambda_n), \quad (11)$$

from which the eigenvalues λ_n are obtained. Finally, the coefficients B_n have to be determined. The roots of equation (11) have the important property that the set of functions

$$J_0(\lambda_i \rho), J_0(\lambda_{i+1} \rho), \ldots$$

are orthogonal on the interval (0, 1) with weight function ρ [9], i.e.

$$\int_0^1 \rho J_0(\lambda_i \rho) J_0(\lambda_j \rho) \mathrm{d}\rho = \begin{bmatrix} 0 & \text{if } i \neq j \\ \dfrac{\lambda_j^2 + Bi^2}{2\lambda_j^2} [J_0(\lambda_j)]^2 & \text{if } i = j \end{bmatrix}. \quad \text{(A8)}$$

Therefore, if the temperature distribution in the source plane with $Z = 1$ is a given function $f(\rho)$ we can write

$$f(\rho) = \sum_{n=1}^{\infty} [B_n J_0(\lambda_n \rho)]$$

and

$$\int_0^1 \rho f(\rho) J_0(\lambda_i \rho) \mathrm{d}\rho = \sum_{n=1}^{\infty} B_n \int_0^1 \rho J_0(\lambda_i \rho) J_0(\lambda_n \rho) \mathrm{d}\rho. \quad \text{(A9)}$$

From equation (A8) with $n = i$ it follows that

$$\int_0^1 \rho f(\rho) J_0(\lambda_n \rho) \mathrm{d}\rho = B_n \frac{\lambda_n^2 + Bi^2}{2\lambda_n^2} [J_0(\lambda_n)]^2. \quad \text{(A10)}$$

For a simple function $f(\rho)$ the integral can usually be solved directly. In particular, if $f(\rho) = 1$ then

$$\int_0^1 \rho J_0(\lambda_n \rho) \mathrm{d}\rho = \frac{J_1(\lambda_n)}{\lambda_n}. \quad \text{(A11)}$$

After rearrangement and using equation (11) the coefficients become

$$B_n = \frac{2Bi}{(\lambda_n^2 + Bi^2) J_0(\lambda_n)}. \quad \text{(A12)}$$

These are inserted in equation (A6) to give the final solution equation (10).

CONDUCTION THERMIQUE TRANSITOIRE DANS LES SYSTEMES CYLINDRIQUES AVEC UNE FRONTIERE EN MOUVEMENT AXIAL

Résumé—Un matériau granuleux chaud est refroidi dès son introduction dans un récipient cylindrique. Le flux thermique au travers de l'enveloppe dépend des matériaux en présence et de leurs températures, de la géométrie du système, aussi bien que de la vitesse de remplissage. Du fait du mouvement de la frontière, il n'existe aucune solution directe de la distribution de température. Cependant, à partir d'un modèle basé sur le principe de l'état quasi-stationnaire, il est possible de calculer la température de la charge et le flux thermique perdu. La théorie est également applicable à des liquides visqueux avec convection interne négligeable. Les résultats sont en bon accord avec ceux fournis par l'expérience et deux applications industrielles sont discutées.

INSTATIONÄRE WÄRMELEITUNG IN ZYLINDERFÖRMIGEN SYSTEMEN MIT EINER SICH AXIAL VERSCHIEBENDEN GRENZE

Zusammenfassung—Beim Füllen zylindrischer Silos mit heissem Granulat kühlt sich dieses bereits während des Füllvorgangs ab. Der Wärmeverlust durch die Seitenwände hängt von den beteiligten Materialien und deren Temperaturen ab, aber auch von der Geometrie des Silos und der Füllgeschwindigkeit. Da sich die obere Systemgrenze mit der Zeit verschiebt, erscheint eine direkte analytische Lösung der Wärmeleitgleichung fraglich. Basierend auf dem Prinzip des quasi-stationären Zustands wird ein Modell entwickelt, das die Berechnung der Temperaturverteilung im Silo gestattet. Die theoretischen Ergebnisse stimmen gut mit experimentellen Beobachtungen überein und sind auch auf zähe Flüssigkeiten bei vernachlässigbarer Konvektion anwendbar. Zur Veranschaulichung werden zwei industrielle Beispiele diskutiert.

НЕСТАЦИОНАРНАЯ ТЕПЛОПРОВОДНОСТЬ В ЦИЛИНДРИЧЕСКИХ СИСТЕМАХ С ПЕРЕМЕЩАЮЩЕЙСЯ ПО ОСИ ГРАНИЦЕЙ

Аннотация — Горячий гранулированный материал охлаждается уже при загрузке в цилиндрический контейнер. Тепловые потери через стенку зависят от теплофизических свойств материалов, температуры, геометрии системы, а также скорости загрузки. В силу подвижности границы отсутствует прямое решение для распределения температуры. Однако, с помощью модели, основанной на принципе квазистационарного состояния, можно рассчитать температуру загружаемого материала и величину тепловых потерь. Теория справедлива также для случая вязкой жидкости при наличии незначительной внутренней конвекции. Теоретические результаты согласуются с экспериментальными. Рассматриваются два случая промышленного применения.